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MAT141 PVK

Aufgaben mit komplexen Zahlen

(von Vorgängern von Luchsinger)

Generelle Empfehlung: Worksheet 6 + Worksheet 10 !!!

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Determinante und Inverse mit komplexen Zahlen

HS15 Probeprüfung Aufgabe 2

i) Berechne die Determinante von

$$\begin{pmatrix} i & 0 & 1 \\ -1 & 1 & -1 \\ 1+i & -1+i & 1-i \end{pmatrix}^4 .$$

iii) Bestimme die Nullstellen von $p(z) = z^4 - i$.

Lösung: i) 16 iii) $e^{\pi/8+(j-1)\cdot\pi/2}$, $1 \leq j \leq 4$

HS18 Uebungen Serie 6 Aufgabe 4b

Berechne die Determinante für folgende Matrizen. Gib an, ob sie regulär sind und falls ja, berechne die Inverse.

$$A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \qquad B = \begin{pmatrix} 1+i & 1 \\ i & 1-i \end{pmatrix} \qquad C = \begin{pmatrix} 1 & i & 2 \\ 2 & 1+2i & 7 \\ 1 & 1+i & 6 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & i & 2 \\ 2 & 1+2i & 8 \\ 2 & 1+i & 6 \end{pmatrix}$$

Lösung: $\det(A) = 0$ $\quad \det(B) = 2 - i \to B^{-1} = \frac{1}{2-i}\begin{pmatrix} 1-i & -1 \\ -i & 1+i \end{pmatrix}$

$\det(C) = 1 \to C^{-1} = \begin{pmatrix} -1+5i & 2-4i & -2+3i \\ -5 & 4 & -3 \\ 1 & -1 & 1 \end{pmatrix}$ $\quad \det(D) = 0$

Lösung HS15 -Probeprüfung - Aufgabe 2

HS15 - Probeprüfung - Aufgabe 2a)

(i) Berechne die Determinante von

$$\begin{pmatrix} i & 0 & 1 \\ -1 & 1 & -1 \\ 1+i & -1+i & 1-i \end{pmatrix}^4 .$$

///// By expanding $\det(A)$ with respect to the first row one computes $\det(A) = i((1-i) + (-1+i)) + 1(1-i-1-i) = -2i$ and $\det(A^4) = \det(A)^4 = 16$. /////

HS15 - Probeprüfung - Aufgabe 2c)

(iii) Bestimme die Nullstellen von $p(z) = z^4 - i$.

///// The solutions of $z^4 = i = e^{i\frac{\pi}{2}}$ are given by

$$z_i = e^{i\psi_j},\ j \in \{0,\dots,3\} \quad \text{with} \quad \psi_j = \frac{\pi}{8} + j(\frac{\pi}{2}). \quad /////$$

Lösung HS18 - Serie 6 - Aufgabe 4 - Fortsetzung

The system has a free parameter, say z_3. Then, we get that

$$z_2 = \frac{1-2i+3z_3}{4+2i}$$

and

$$2z_1 = i + (2+i)z_2 - z_3 = i + (1-2i+3z_3)/2 - z_3 = \frac{1+z_3}{2}$$

so that the set of solutions is

$$L = \left\{ \left(\frac{1+z_3}{4}, \frac{1-2i+3z_3}{4+2i}, z_3 \right), z_3 \in \mathbb{C} \right\}.$$

(iv) The extended coefficient matrix is

$$\left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 4i & -7 & 0 \\ 0 & -5 & -9i & -1 \end{array}\right) \overset{\substack{II\to 5II \\ III\to 4iIII}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 20i & -35 & 0 \\ 0 & -20i & 36 & -1 \end{array}\right) \overset{III+II}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 20i & -35 & 0 \\ 0 & 0 & 1 & -4i \end{array}\right)$$

which yields the set of solutions

$$L = \{(7i, -7, -4i)\}.$$

(b) We have $\det A = i(-i) - 1 = 0$, hence this matrix is not regular. Then, $\det B = (1+i)(1-i) - i = 2-i$. Hence, B is invertible, and we can use the usual formula for the inverse and

$$B^{-1} = \frac{1}{2-i}\begin{pmatrix} 1-i & -1 \\ -i & 1+i \end{pmatrix}.$$

For the 3×3 matrices we use row operations. For C, we have

$$C = \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 7 & 0 & 1 & 0 \\ 1 & 1+i & 6 & 0 & 0 & 1 \end{array}\right) \overset{\substack{II-2I \\ III-I}}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 1 & 4 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{III-II}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

so C is regular and the determinant of C is 1. Continuing, we get

$$\left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \overset{\substack{II-3III \\ I-2III}}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & i & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & -5 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\overset{I-iII}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1+5i & 2-4i & -2+3i \\ 0 & 1 & 0 & -5 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

so the inverse of C is

$$C^{-1} = \begin{pmatrix} -1+5i & 2-4i & -2+3i \\ -5 & 4 & -3 \\ 1 & -1 & 1 \end{pmatrix}.$$

Similarly, for D we have

$$D = \left(\begin{array}{ccc|ccc} 0 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 8 & 0 & 1 & 0 \\ 2 & 1+i & 6 & 0 & 0 & 1 \end{array}\right) \overset{III\leftrightarrow I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 8 & 0 & 1 & 0 \end{array}\right)$$

$$\overset{III-I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 0 & i & 2 & 0 & 1 & -1 \end{array}\right) \overset{III-II}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right)$$

hence D is not regular, i.e. $\det(D) = 0$.

LGS mit komplexen Zahlen

HS18 Uebungen Serie 6 Aufgabe 4a)

Löse folgendes komplexe lineare Gleichungssystem:

(i)
$$\begin{aligned} z_1 + iz_2 &= 2 \\ (1+i)z_1 + (i-1)z_2 &= 2i+2 \end{aligned}$$

(ii)
$$\begin{aligned} iz_1 + z_2 &= 1 \\ z_1 + iz_2 &= i \end{aligned}$$

(iii)
$$\begin{aligned} 2z_1 - (2+i)z_2 + z_3 &= i \\ 4z_1 - iz_3 &= 1 \end{aligned}$$

(iv)
$$\begin{aligned} z_1 + iz_2 &= 0 \\ 4iz_2 - 7z_3 &= 0 \\ -5z_2 - 9iz_3 &= 1 \end{aligned}$$

Lösung: (i) $L = \{(2 - iz_2, z_2),\ z_2 \in \mathbb{C}\}$ (ii) $L = \{(0,1)\}$
(iii) $L = \{(\frac{1+z_3}{4}, \frac{1-2i+3z_3}{4+2i}), z_3\ z_3 \in \mathbb{C}\}$ (iv) $L = \{(7i, -7, -4i)\}$

HS17 Uebungen Serie 6 Aufgabe 4

(a) Für welche Werte von $a \in \mathbb{C}$ hat das folgende komplexe lineare Gleichungssystem

(i) keine Lösung? (ii) genau eine Lösung? (iii) unendlich viele Lösungen?

$$\begin{aligned} z_1 + iz_2 + (1-i)z_3 &= 0 \\ ia^2 z_1 + z_2 + (1+i)a^2 z_3 &= a - i \\ iz_1 - z_2 + iz_3 &= a + i \end{aligned}$$

(b) Lösen Sie das Gleichungssystem für $a = 3i$.

Lösung: (a) (i) $a = -i$ (ii) $a \neq \pm i$ (iii) $a = +i$ (b) $L = \{(15/4 + 4i, -i/4, -4i)\}$

Lösung HS18 - Serie 6 - Aufgabe 4

Exercise 4 (Complex linear systems and matrices)

(a) Give the sets of solutions of the following systems of linear equations[1]:

(i)

$$\begin{cases} z_1 + iz_2 & = 2 \\ (1+i)z_1 + (i-1)z_2 & = 2i+2 \end{cases}$$

(ii)

$$\begin{cases} iz_1 + z_2 & = 1 \\ z_1 + iz_2 & = i \end{cases}$$

(iii)

$$\begin{cases} 2z_1 - (2+i)z_2 + z_3 & = i \\ 4z_1 - z_3 & = 1 \end{cases}$$

(iv)

$$\begin{cases} z_1 + iz_2 & = 0 \\ 4iz_2 - 7z_3 & = 0 \\ -5z_2 - 9iz_3 & = 1 \end{cases}$$

(b) Compute the determinant of the following matrices. Decide whether they are regular. If so, compute the inverse matrix.

$$A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \qquad B = \begin{pmatrix} 1+i & 1 \\ i & 1-i \end{pmatrix} \qquad C = \begin{pmatrix} 1 & i & 2 \\ 2 & 1+2i & 7 \\ 1 & 1+i & 6 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & i & 2 \\ 2 & 1+2i & 8 \\ 2 & 1+i & 6 \end{pmatrix}$$

Solution:

(a) Also for linear systems with complex coefficients we can use the Gauss algorithm as before.

(i) The extended coefficient matrix is

$$\left(\begin{array}{cc|c} 1 & i & 2 \\ (1+i) & (i-1) & 2i+2 \end{array}\right) \underset{II-(1+i)I}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & i & 2 \\ 0 & 0 & 0 \end{array}\right)$$

Hence the set of solutions has a free parameter, say z_2, and by equation I we get $z_1 = 2 - iz_2$, so that the set of solutions is

$$L = \{(2 - iz_2, z_2), z_2 \in \mathbb{C}\}.$$

Notice that since we consider complex solutions now, the parameter z_2 can take any *complex value.*

(ii) The extended coefficient matrix is

$$\left(\begin{array}{cc|c} i & 1 & 1 \\ 1 & i & i \end{array}\right) \underset{II+iI}{\rightsquigarrow} \left(\begin{array}{cc|c} i & 1 & 1 \\ 0 & 2i & 2i \end{array}\right)$$

Hence we get $z_2 = i$ and then from the first equation $z_1 = 1$, so the set of solutions is

$$L = \{(0, 1)\}.$$

(iii) The extended coefficient matrix is

$$\left(\begin{array}{ccc|c} 2 & -(2+i) & 1 & i \\ 4 & 0 & -1 & 1 \end{array}\right) \underset{II-2I}{\rightsquigarrow} \left(\begin{array}{ccc|c} 2 & -(2+i) & 1 & i \\ 0 & (4+2i) & -3 & 1-2i \end{array}\right)$$

[1] We now use $z_1, z_2, \ldots$ as variables to emphasize that solutions can be complex

Lösung HS18 - Serie 6 - Aufgabe 4 - Fortsetzung

The system has a free parameter, say z_3. Then, we get that

$$z_2 = \frac{1-2i+3z_3}{4+2i}$$

and

$$2z_1 = i + (2+i)z_2 - z_3 = i + (1-2i+3z_3)/2 - z_3 = \frac{1+z_3}{2}$$

so that the set of solutions is

$$L = \left\{ \left(\frac{1+z_3}{4}, \frac{1-2i+3z_3}{4+2i}, z_3 \right), z_3 \in \mathbb{C} \right\}.$$

(iv) The extended coefficient matrix is

$$\left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 4i & -7 & 0 \\ 0 & -5 & -9i & -1 \end{array}\right) \overset{\substack{II\to 5II \\ III\to 4iIII}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 20i & -35 & 0 \\ 0 & -20i & 36 & -1 \end{array}\right) \overset{III+II}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 20i & -35 & 0 \\ 0 & 0 & 1 & -4i \end{array}\right)$$

which yields the set of solutions

$$L = \{(7i, -7, -4i)\}.$$

(b) We have $\det A = i(-i)-1 = 0$, hence this matrix is not regular. Then, $\det B = (1+i)(1-i)-i = 2-i$. Hence, B is invertible, and we can use the usual formula for the inverse and

$$B^{-1} = \frac{1}{2-i}\begin{pmatrix} 1-i & -1 \\ -i & 1+i \end{pmatrix}.$$

For the 3×3 matrices we use row operations. For C, we have

$$C = \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 7 & 0 & 1 & 0 \\ 1 & 1+i & 6 & 0 & 0 & 1 \end{array}\right) \overset{\substack{II-2I \\ III-I}}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 1 & 4 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{III-II}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

so C is regular and the determinant of C is 1. Continuing, we get

$$\left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \overset{\substack{II-3III \\ I-2III}}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & i & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & -5 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\overset{I-iII}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1+5i & 2-4i & -2+3i \\ 0 & 1 & 0 & -5 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

so the inverse of C is

$$C^{-1} = \begin{pmatrix} -1+5i & 2-4i & -2+3i \\ -5 & 4 & -3 \\ 1 & -1 & 1 \end{pmatrix}.$$

Similarly, for D we have

$$D = \left(\begin{array}{ccc|ccc} 0 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 8 & 0 & 1 & 0 \\ 2 & 1+i & 6 & 0 & 0 & 1 \end{array}\right) \overset{III\leftrightarrow I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 8 & 0 & 1 & 0 \end{array}\right)$$

$$\overset{III-I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 0 & i & 2 & 0 & 1 & -1 \end{array}\right) \overset{III-II}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right)$$

hence D is not regular, i.e. $\det(D) = 0$.

Lösung HS17 - Serie 6 - Aufgabe 4

Exercise 4. (4 points) Consider the following linear system with complex coefficients

$$
\begin{aligned}
&z_1 + iz_2 + (1-i)z_3 = 0 \\
&ia^2 z_1 + z_2 + (1+i)a^2 z_3 = a - i \\
&iz_1 - z_2 + iz_3 = a + i\,,
\end{aligned}
$$

where $a \in \mathbb{C}$.

1. Use Gaussian elimination to bring it in row echelon form.(1 point)
2. Discuss the existence and unicity of solutions according to the values of a. (2 points)
3. Solve the system explicitly for $a = 3i$. (1 point)

Solution.

1. By substituting $R_2 \to R_2 - ia^2 R_1$ and $R_3 \to R_3 - iR_1$ we find

$$\left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ ia^2 & 1 & (1+i)a^2 & a-i \\ i & -1 & i & a+i \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ 0 & 1+a^2 & 0 & a-i \\ 0 & 0 & -1 & a+i \end{array}\right).$$

2. For $a^2+1 \neq 0 \leftrightarrow a \neq \pm i$ the coefficient matrix is non singular, hence the solution is unique.
   For $a = -i$ the system becomes

$$\left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 0 & -1 & 0 \end{array}\right).$$

   The second equation reads $0 \cdot z_2 = -2i$, so the system has no solution.
   For $a = i$ we have

$$\left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2i \end{array}\right).$$

   The second equation gives $0 \cdot z_2 = 0$, thus the system has infinite solutions.

3. For $a = 3i$ the system becomes

$$\left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ 0 & -8 & 0 & 2i \\ 0 & 0 & -1 & 4i \end{array}\right)$$

   and the solution is given by

$$L = \left\{ \left(\frac{15}{4} + 4i, -\frac{i}{4}, -4i \right) \right\}. \qquad \blacksquare$$

linear unabhängig / Basis mit komplexen Zahlen

HS15 Uebungen Serie 6 Aufgabe 1

a) Entscheide, ob folgende Vektoren in $\mathbb{C}^3$ linear abhängig sind oder nicht:

$$a^{(1)} = \begin{bmatrix} 1+i \\ 1-i \\ -1+i \end{bmatrix}, \qquad a^{(2)} = \begin{bmatrix} 0 \\ 4-2i \\ -3+5i \end{bmatrix}, \qquad a^{(3)} = \begin{bmatrix} 1+i \\ 0 \\ 3 \end{bmatrix}.$$

b) Für welche komplexen Zahlen $z \in \mathbb{C}$ verschwindet die Determinante von

$$\begin{bmatrix} 1+2i & 3+4i \\ z & 1-2i \end{bmatrix}^{25}$$

Lösung: a) linear unabhängig b) $z = \frac{3}{5} - \frac{4}{5}i$

HS17 Uebungen Serie 7 Aufgabe 1

Entscheide, ob folgende Vektoren in $\mathbb{C}^2$ eine Basis von $\mathbb{C}^2$ bilden:

(a)

$$v^{(1)} = \begin{pmatrix} 1+i \\ -1+i \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} -2+2i \\ -2-2i \end{pmatrix}$$

(b)

$$v^{(1)} = \begin{pmatrix} 1+i \\ 2-i \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} -i \\ 2+2i \end{pmatrix}$$

Lösung:

(a) $\det(v^{(1)}, v^{(2)}) = 0 \rightarrow$ linear abhängig $\Rightarrow$ keine Basis

(b) $\det(v^{(1)}, v^{(2)}) = 1 + 6i \neq 0 \rightarrow$ linear unabhängig $\Rightarrow$ Basis

Lösung HS15 - Serie 6 - Aufgabe 1

(i) Entscheide, ob folgende Vektoren in $\mathbb{C}^3$ linear abhängig sind oder nicht:

$$\text{(i1)} \quad a^{(1)} = \begin{bmatrix} 1 \\ 2+\mathrm{i} \\ 1 \end{bmatrix}, \qquad a^{(2)} = \begin{bmatrix} -1+3\mathrm{i} \\ 1+\mathrm{i} \\ 3+\mathrm{i} \end{bmatrix}.$$

///// The first step of Gauss elimination yields

$$\begin{bmatrix} 1 & -1+3\mathrm{i} \\ 2+\mathrm{i} & 1+\mathrm{i} \\ 1 & 3+\mathrm{i} \end{bmatrix} \to \begin{bmatrix} 1 & -1+3\mathrm{i} \\ 0 & (1+\mathrm{i})-(2+\mathrm{i})(-1+3\mathrm{i}) \\ 0 & 4-2\mathrm{i} \end{bmatrix} = \begin{bmatrix} 1 & -1+3\mathrm{i} \\ 0 & (1+\mathrm{i})-(-5+5\mathrm{i}) \\ 0 & 4-2\mathrm{i} \end{bmatrix} = \begin{bmatrix} 1 & -1+3\mathrm{i} \\ 0 & 6-4\mathrm{i} \\ 0 & 4-2\mathrm{i} \end{bmatrix},$$

hence the two vectors are $\mathbb{C}$-linearly independent. /////

$$\text{(i2)} \quad a^{(1)} = \begin{bmatrix} 1+\mathrm{i} \\ 1-\mathrm{i} \\ -1+\mathrm{i} \end{bmatrix}, \qquad a^{(2)} = \begin{bmatrix} 0 \\ 4-2\mathrm{i} \\ -3+5\mathrm{i} \end{bmatrix}, \qquad a^{(3)} = \begin{bmatrix} 1+\mathrm{i} \\ 0 \\ 3 \end{bmatrix}.$$

///// The first step of Gauss elimination yields

$$\begin{bmatrix} 1+\mathrm{i} & 0 & 1+\mathrm{i} \\ 1-\mathrm{i} & 4-2\mathrm{i} & 0 \\ -1+\mathrm{i} & -3+5\mathrm{i} & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4-2\mathrm{i} & -1+\mathrm{i} \\ 0 & -3+5\mathrm{i} & 4-\mathrm{i} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 20-10\mathrm{i} & -5+5\mathrm{i} \\ 0 & -6+10\mathrm{i} & 8-2\mathrm{i} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 14 & 3+3\mathrm{i} \\ 0 & -6+10\mathrm{i} & 8-2\mathrm{i} \end{bmatrix}$$

and we compute

$$8-2\mathrm{i}-\frac{1}{14}(3+3\mathrm{i})(-6+10\mathrm{i}) = 8-2\mathrm{i}-\frac{1}{14}(-48+12\mathrm{i}) \neq 0.$$

Therefore, the three vectors are $\mathbb{C}$-linearly independent.

Alternatively, one can compute the determinant

$$(1+\mathrm{i})(4-2\mathrm{i})3+(1+\mathrm{i})(1-\mathrm{i})(-3+5\mathrm{i})-((-1+\mathrm{i})(4-2\mathrm{i})(1+\mathrm{i}))$$
$$= (6+2\mathrm{i})3+2(-3+5\mathrm{i})+2(4-2\mathrm{i}) = (18-6+8)+(6+10-4)\mathrm{i} = 20+12\mathrm{i} \neq 0. \quad /////$$

(ii) Für welche komplexen Zahlen $z \in \mathbb{C}$ verschwindet die Determinante von

$$B = \begin{bmatrix} 1+2\mathrm{i} & 3+4\mathrm{i} \\ z & 1-2\mathrm{i} \end{bmatrix}^{25}$$

///// Clearly,

$$\det B = [(1+2\mathrm{i})(1-2\mathrm{i})-(3+4\mathrm{i})z]^{25} = [5-(3+4\mathrm{i})z]^{25}.$$

Therefore, $\det B = 0$ if

$$z = \frac{5}{3+4\mathrm{i}} = \frac{5(3-4\mathrm{i})}{(3+4\mathrm{i})(3-4\mathrm{i})} = \frac{15-20\mathrm{i}}{25} = \frac{3}{5}-\frac{4}{5}\mathrm{i}. \quad /////$$

Lösung HS17 - Serie 7 - Aufgabe 1

Exercise 1 (4 points). Decide if the given vectors in $\mathbb{C}^2$ form a basis of $\mathbb{C}^2$.

1.
$$v^{(1)} = \begin{pmatrix} 1+\mathrm{i} \\ -1+\mathrm{i} \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} -2+2\mathrm{i} \\ -2-2\mathrm{i} \end{pmatrix}.$$

2.
$$v^{(1)} = \begin{pmatrix} 1+\mathrm{i} \\ 2-\mathrm{i} \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} -\mathrm{i} \\ 2+2\mathrm{i} \end{pmatrix}.$$

Solution.

1. $v^{(2)} = 2\mathrm{i} \cdot v^{(1)}$. Because the two vectors are dependent, the do not form a basis.
2. $\det \left(v^{(1)}\ v^{(2)}\right) = 1 + 6\mathrm{i} \neq 0$. Therefore, the two vectors are independent and form a basis.

Eigenwerte und Eigenvektoren mit komplexen Zahlen

HS15 Uebungen Serie 9 Aufgabe 3

i) Bestimme die Eigenwerte und dazugehörigen Eigenvektoren von $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Finde eine Basis $[v] = [v^{(1)}, v^{(2)}]$ von $\mathbb{C}^2$ so dass $(T_A)_{[v]\to[v]}$ eine Diagonalmatrix ist.

ii) Dieselbe Aufgabe wie in i) für die Matrix $B = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}$.

Lösung:

i) $\lambda_1 = 1-2i,\ \lambda_2 = 1+2i \quad [v] = [v^{(1)}, v^{(2)}] = [(i,1)^T, (-i,1)^T] \quad D = \begin{pmatrix} 1-2i & 0 \\ 0 & 1+2i \end{pmatrix}$

ii) $z_1 = -1,\ z_2 = 6 \quad [v] = [v^{(1)}, v^{(2)}] = [(-\frac{3+i}{2}, 1)^T, (\frac{3+i}{5}, 1)^T] \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$

HS15 Uebungen Serie 9 Aufgabe 4

Bestimme die Eigenwerte und Eigenräume von folgenden Matrizen.

$$C = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -5 \\ 0 & 1 & -2 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

Lösung:

c) EW: $z_1 = \cos(\varphi) + i\sin(\varphi),\ z_2 = \cos(\varphi) - i\sin(\varphi) \quad E_{z_1}(C) = \{a(i,1), a \in \mathbb{C}\}, E_{z_2}(C) = \{a(-i,1), a \in \mathbb{C}\}$

d) EW: $z_1 = 3,\ z_2 = -i,\ z_3 = i \quad E_3(D) = \{a(1,0,0), a \in \mathbb{C}\}, E_{-i}(D) = \{a(0,5,2+i), a \in \mathbb{C}\}, E_i(D) = \{a(0,5,2-i), a \in \mathbb{C}\}$

f) EW: $z_1 = i,\ z_2 = -i \quad E_i(F) = \{a(1+i,2), a \in \mathbb{C}\}, E_{-i}(F) = \{a(1-i,2), a \in \mathbb{C}\}$

Lösung HS15 - Serie 9 - Aufgabe 3

(i) Bestimme die Eigenwerte und dazugehörige Eigenvektoren von $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Finde eine Basis $[v] = [v^{(1)}, v^{(2)}]$ von $\mathbb{C}^2$ so dass $(T_A)_{[v]\to[v]}$ eine Diagonalmatrix ist.

///// We compute

$$\det(A - z) = (1-z)^2 + 4 = z^2 - 2z + 5.$$

Consequently, the eigenvalues are given by

$$z_\pm = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2\mathrm{i}.$$

To determine the eigenvectors we solve $(A - z_\pm)x = 0$. That is for z_-

$$\left[\begin{array}{cc|c} 2\mathrm{i} & 2 & 0 \\ -2 & 2\mathrm{i} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2\mathrm{i} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

hence we may choose $v_- = (\mathrm{i}, 1)$. Similarly, for z_+,

$$\left[\begin{array}{cc|c} -2\mathrm{i} & 2 & 0 \\ -2 & -2\mathrm{i} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -2\mathrm{i} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so we may choose $v_+ = (-\mathrm{i}, 1)$. Clearly, v_+ and v_- are $\mathbb{C}$-linearly independent. Let $[v] = [v_-, v_+]$ and note that $Av_\pm = z_\pm v_\pm$ hence

$$(T_A)_{[v]\to[v]} = \begin{bmatrix} 1-2\mathrm{i} & 0 \\ 0 & 1+2\mathrm{i} \end{bmatrix}. \quad /////$$

(ii) Dieselbe Aufgabe wie in (i) für die Matrix $B = \begin{bmatrix} 1 & 3+\mathrm{i} \\ 3-\mathrm{i} & 4 \end{bmatrix}$.

///// We compute the characteristic polynomial to be

$$\det(B - z) = (1-z)(4-z) - (3+\mathrm{i})(3-\mathrm{i}) = z^2 - 5z - 6.$$

Therefore,

$$z_\pm = \frac{5 \pm \sqrt{49}}{2} = \frac{5 \pm 7}{2}.$$

The eigenvalues thus are $z_- = -1$ and $z_+ = 6$. To compute the eigenvectors we solve $(A - z_\pm)x = 0$. So for z_-

$$\left[\begin{array}{cc|c} 2 & 3+\mathrm{i} & 0 \\ 3-\mathrm{i} & 5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3+\mathrm{i} & 0 \\ 2 & 3+\mathrm{i} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3+\mathrm{i} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

hence $v_- = (-\frac{3+\mathrm{i}}{2}, 1)$, and further for z_+

$$\left[\begin{array}{cc|c} -5 & 3+\mathrm{i} & 0 \\ 3-\mathrm{i} & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -5 & 3+\mathrm{i} & 0 \\ -5 & 3+\mathrm{i} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -5 & 3+\mathrm{i} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

we get $v_+ = (\frac{3+\mathrm{i}}{5}, 1)$. With $[v] = [v_-, v_+]$ we obtain

$$(T_B)_{[v]\to[v]} = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}. \quad /////$$

Lösung HS15 - Serie 9 - Aufgabe 4

(iii) $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$.

///// Since $\det(A - z) = z^2 - (2\cos\varphi)z + 1$, we compute for the eigenvalues

$$z_\pm = \frac{2\cos\varphi \pm \sqrt{4\cos^2\varphi - 4}}{2} = \cos\varphi \pm \mathrm{i}\sin\varphi.$$

To compute the eigenvectors we consider

$$A - z_\pm = \begin{bmatrix} \mp\mathrm{i}\sin\varphi & -\sin\varphi \\ \sin\varphi & \mp\mathrm{i}\sin\varphi \end{bmatrix} = \sin\varphi \begin{bmatrix} \mp\mathrm{i} & -1 \\ 1 & \mp\mathrm{i} \end{bmatrix} \to \sin\varphi \begin{bmatrix} 1 & \mp\mathrm{i} \\ 0 & 0 \end{bmatrix},$$

hence $v_\pm = (\pm\mathrm{i}, 1)$ so

$$E_{\cos\varphi - \mathrm{i}\sin\varphi} = \{\alpha(-\mathrm{i}, 1) : \alpha \in \mathbb{C}\}, \qquad E_{\cos\varphi + \mathrm{i}\sin\varphi} = \{\alpha(\mathrm{i}, 1) : \alpha \in \mathbb{C}\}.$$ /////

(iv) $$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -5 \\ 0 & 1 & -2 \end{bmatrix}$$

///// A straightforward computation shows

$$\det(A - z) = (3 - z)((2 - z)(-2 - z) + 5) = (3 - z)(z^2 + 1) = (3 - z)(z - \mathrm{i})(z + \mathrm{i}).$$

So the eigenvalues are $z_1 = 3$, $z_2 = -\mathrm{i}$ and $z_3 = \mathrm{i}$. Clearly, $v_1 = (1, 0, 0)$ is an eigenvector for z_1 and

$$E_3(A) = \{\alpha(1, 0, 0) : \alpha \in \mathbb{C}\}.$$

To proceed, consider

$$\begin{bmatrix} 3 + \mathrm{i} & 0 & 0 \\ 0 & 2 + \mathrm{i} & -5 \\ 0 & 1 & -2 + \mathrm{i} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 + \mathrm{i} & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

hence $v_2 = (0, 5, 2 + \mathrm{i})$ and

$$E_{-\mathrm{i}}(A) = \{\alpha(0, 5, 2 + \mathrm{i}) : \alpha \in \mathbb{C}\}.$$

Similarly,

$$\begin{bmatrix} 3 - \mathrm{i} & 0 & 0 \\ 0 & 2 - \mathrm{i} & -5 \\ 0 & 1 & -2 - \mathrm{i} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 - \mathrm{i} & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

gives $v_3 = (0, 5, 2 - \mathrm{i})$ and

$$E_{\mathrm{i}}(A) = \{\alpha(0, 5, 2 - \mathrm{i}) : \alpha \in \mathbb{C}\}.$$ /////

(vi) * (bonus)

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

///// A straightforward computation shows

$$\det(A - z) = (1 - z)(-1 - z) + 2 = z^2 + 1 = (z + \mathrm{i})(z - \mathrm{i}),$$

so the eigenvalues are $z_\pm = \pm\mathrm{i}$. To compute the eigenvectors we consider

$$\begin{bmatrix} 1 \mp \mathrm{i} & -1 \\ 2 & -1 \mp \mathrm{i} \end{bmatrix} \to \begin{bmatrix} 2 & -1 \mp \mathrm{i} \\ 0 & 0 \end{bmatrix}$$

hence $v_\pm = (1 \pm \mathrm{i}, 2)$ and

$$E_{\pm\mathrm{i}}(A) = \{\alpha(1 \pm \mathrm{i}, 2) : \alpha \in \mathbb{C}\}.$$ /////

Diagonalisieren mit komplexen Zahlen

HS15 Uebungen Serie 10 Aufgabe 1

Sei $A = \begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix} \in \mathbb{C}^{2x2}$.

i) Verifiziere, dass A hermitesch ist.

ii) Bestimme die Eigenwerte von A.

iii) Finde eine unitäre Matrix S so dass $S^{-1}AS$ eine Diagonalmatrix ist.

Lösung:

i) $\overline{A^T} = A$ ii) $\lambda_1 = -1, \lambda_2 = 3$

iii) $S = \frac{1}{\sqrt{2}}[v^{(1)}\ v^{(2)}]$ mit $v^{(1)} = (-i, 1)^T,\ v^{(2)} = (i, 1)^T$ und $D = \text{diag}(-1, 3)$

HS15 Uebungen Serie 10 Aufgabe 2

Sei A die 3 x 3 Matrix

$$A == \begin{bmatrix} 2 & i & 1 \\ -i & 2 & -i \\ 1 & i & 2 \end{bmatrix}.$$

i) Verifiziere, dass A hermitesch ist.

ii) Bestimme die Eigenwerte von A.

iii) Finde eine unitäre Matrix S so dass $S^{-1}AS$ eine Diagonalmatrix ist.

Lösung:

i) $\overline{A^T} = A$ ii) $\lambda_1 = 1 = \lambda_2, \lambda_3 = 4$

iii) $S = \frac{1}{\sqrt{2}}[v^{(1)}\ v^{(2)}\ v^{(3)}]$ mit $v^{(1)} = \frac{1}{\sqrt{2}}(-i, 1, 0)^T$,

$v^{(2)} = \frac{1}{\sqrt{3/2}}(-1/2, i/2, 1)^T,\ v^{(3)} = \frac{1}{\sqrt{3}}(1, -i, 1)^T$

HS18 Uebungen Serie 13 Aufgabe 4

Consider the matrix below

$$A = \begin{bmatrix} 3 & -i & 0 \\ i & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) Without computation, argue that A is diagonalizabe.

b) Find all eigenvalues and corresponding eigenspaces of A.

c) Find a unitary matrix U such that $UAU^{-1} = D$ is a diagonal matrix.

Lösung:

a) $A^T = \overline{A}$ b) $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 4$

$E_A(1) = \{(0, 0, x_3)^T | x_3 \in \mathbb{C}\}, \quad E_A(2) = \{(ix_2, x_2, 0)^T | x_2 \in \mathbb{C}\}, \quad E_A(4) = \{(-ix_2, x_2, 0)^T | x_2 \in \mathbb{C}\}$

c) $v_1 = (0, 0, 1)^T, v_2 = \left(\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T, v_3 = \left(\frac{-i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T \quad U = (v_1\ v_2\ v_3)$

Lösung HS15 - Serie 10 - Aufgabe 1

Aufgabe 1 Sei $A = \begin{bmatrix} 1 & 2\mathrm{i} \\ -2\mathrm{i} & 1 \end{bmatrix} \in \mathbb{C}^{2\times 2}$.

(i) Verifiziere, dass A hermitesch ist.

///// $$\overline{A^\top} = \begin{bmatrix} 1 & 2\mathrm{i} \\ -2\mathrm{i} & 1 \end{bmatrix} = A.$$ /////

(ii) Bestimme die Eigenwerte von A.

///// We note that $\chi(z) = (1-z)^2 - 4 = z^2 - 2z - 3$ and further compute

$$z_\pm = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2.$$

Hence $\lambda_1 = -1$ and $\lambda_2 = 3$ are the eigenvalues of A. /////

(iii) Finde eine unitäre Matrix S so dass $S^{-1}AS$ eine Diagonalmatrix ist.

///// Since A is hermitean it suffices to compute the eigenvectors. For λ_1

$$\begin{bmatrix} 2 & 2\mathrm{i} \\ -2\mathrm{i} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \mathrm{i} \\ 0 & 0 \end{bmatrix},$$

hence $v^{(1)} = (-\mathrm{i}, 1)$. For λ_2

$$\begin{bmatrix} -2 & 2\mathrm{i} \\ -2\mathrm{i} & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\mathrm{i} \\ 0 & 0 \end{bmatrix},$$

hence $v^{(2)} = (\mathrm{i}, 1)$. Clearly $\|v^{(1)}\| = \|v^{(2)}\| = \sqrt{2}$ so let $S = \frac{1}{\sqrt{2}}[v^{(1)}\ v^{(2)}]$ then

$$S^{-1}AS = S^*AS = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$ /////

Lösung HS15 - Serie 10 - Aufgabe 2

$$A = \begin{bmatrix} 2 & \mathrm{i} & 1 \\ -\mathrm{i} & 2 & -\mathrm{i} \\ 1 & \mathrm{i} & 2 \end{bmatrix}.$$

(i) Verifiziere, dass A hermitesch ist.

///// $$\overline{A^\top} = \begin{bmatrix} 2 & \mathrm{i} & 1 \\ -\mathrm{i} & 2 & -\mathrm{i} \\ 1 & \mathrm{i} & 2 \end{bmatrix} = A.$$ /////

(ii) Bestimme die Eigenwerte von A.

///// We compute

$$\begin{aligned}\chi_A(z) &= (2-z)^3 + 1 + 1 - ((2-z) + (2-z) + (2-z)) \\ &= (2-z)^3 - 3(2-z) + 2 = -z^3 + 6z^2 - 12z + 8 + 3z - 4 \\ &= -z^3 + 6z^2 - 9z + 4.\end{aligned}$$

Clearly, $\lambda_1 = 1$ is a root of χ_A and we compute

$$\frac{-z^3 + 6z^2 - 9z + 4}{z-1} = -z^2 + 5z - 4 = -(z-1)(z-4).$$

Therefore $\lambda_2 = 1$ and $\lambda_3 = 4$. /////

(iii) Bestimme eine unitäre Matrix S so dass $S^{-1}AS$ eine Diagonalmatrix ist.

///// We compute the eigenvectors of A. For $\lambda_1 = \lambda_2$

$$\begin{bmatrix} 1 & \mathrm{i} & 1 \\ -\mathrm{i} & 1 & -\mathrm{i} \\ 1 & \mathrm{i} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \mathrm{i} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence $v^{(1)} = (-\mathrm{i}, 1, 0)$ and $v^{(2)} = (-1, 0, 1)$. Note that $v^{(1)}$ and $v^{(2)}$ are not orthogonal. Indeed $\langle v^{(2)}, v^{(1)} \rangle = -\mathrm{i} \neq 0$. However, by setting

$$\tilde{v}^{(2)} = v^{(2)} - \frac{\langle v^{(2)}, v^{(1)} \rangle}{\langle v^{(1)}, v^{(1)} \rangle} v^{(1)} = v^{(2)} + \frac{\mathrm{i}}{2} v^{(1)} = (-1/2, \mathrm{i}/2, 1),$$

we find

$$\langle \tilde{v}^{(2)}, v^{(1)} \rangle = \langle v^{(2)}, v^{(1)} \rangle - \frac{\langle v^{(2)}, v^{(1)} \rangle}{\langle v^{(1)}, v^{(1)} \rangle} \langle v^{(1)}, v^{(1)} \rangle = 0.$$

For λ_3 we compute

$$\begin{bmatrix} -2 & \mathrm{i} & 1 \\ -\mathrm{i} & -2 & -\mathrm{i} \\ 1 & \mathrm{i} & -2 \end{bmatrix} \to \begin{bmatrix} -2 & \mathrm{i} & 1 \\ 2 & -4\mathrm{i} & 2 \\ 2 & 2\mathrm{i} & -4 \end{bmatrix} \to \begin{bmatrix} -2 & \mathrm{i} & 1 \\ 0 & -3\mathrm{i} & 3 \\ 0 & 3\mathrm{i} & -3 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & \mathrm{i} \\ 0 & 0 & 0 \end{bmatrix},$$

so $v^{(3)} = (1, -\mathrm{i}, 1)$. Since $\|v^{(1)}\| = \sqrt{2}$, $\|\tilde{v}^{(2)}\| = \sqrt{3/2}$, and $\|v^{(3)}\| = \sqrt{3}$, let

$$S = \begin{bmatrix} -\frac{\mathrm{i}}{\sqrt{2}} & \frac{-1/2}{\sqrt{3/2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{\mathrm{i}/2}{\sqrt{3/2}} & -\frac{\mathrm{i}}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3/2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

then $S^* = S^{-1}$ and

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$ /////

Lösung HS18 Serie 13 Aufgabe 4

Exercise 4 (Diagonalization)
Consider the matrix below:

$$A = \begin{pmatrix} 3 & -i & 0 \\ i & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Without computation, argue that A is diagonalizable.

(b) Find all the eigenvalus and corresponding eigenspaces of A.

(c) Find a unitary matrix U such that $UAU^{-1} = D$ is a diagonal matrix.

Solution:

(a) The matrix A is Hermitian since $A^T = \overline{A}$. Hence A is diagonalzable and there exists an orthonormal basis of eigenvectors.

(b) We compute the characteristic polynomial:

$$P_A(z) = \det \begin{pmatrix} 3-z & -i & 0 \\ i & 3-z & 0 \\ 0 & 0 & 1-z \end{pmatrix} = (1-z)[(3-z)^2 - 1].$$

Hence the first eigenvalue is given by $\lambda_1 = 1$. The other eigenvalues are given by the zeroes of the polynomial

$$(3-z)^2 - 1 = z^2 - 6z + 8 = (z-2)(z-4)$$

hence they are $\lambda_2 = 2$ and $\lambda_3 = 4$. We proceed to compute the eigenspaces. The eigenspace to λ_1 is given by

$$E_A(1) = \ker(A - I) = \ker \begin{pmatrix} 2 & -i & 0 \\ i & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \{(0, 0, x_3) | x_3 \in \mathbb{C}\}.$$

The eigenspace to λ_2 is given by

$$E_A(2) = \ker(A - 2I) = \ker \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \{(x_1, x_2, 0)^T | x_1 - ix_2 = 0\} = \{(ix_2, x_2, 0)^T | x_2 \in \mathbb{C}\}.$$

The eigenspace to λ_3 is given by

$$E_A(4) = \ker(A - 4I) = \ker \begin{pmatrix} -1 & -i & 0 \\ i & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \{(x_1, x_2, 0)^T | -x_1 - ix_2 = 0\} = \{(-ix_2, x_2, 0)^T | x_2 \in \mathbb{C}\}.$$

(c) We have to find a normalized vector in every eigenspace. Such vectors are given by $v_1 = (0, 0, 1)^T$, $v_2 = \frac{1}{\sqrt{2}}(i, 1, 0)^T$, $v_3 = \frac{1}{\sqrt{2}}(-i, 1, 0)^T$. The unitary matrix that we are looking for is then $U = (v_1 | v_2 | v_3)$.

Zeige dass / Prüfe ob mit komplexen Zahlen

MAT141 HS17 Serie 10 Aufgabe 3:

Für welche Werte des komplexen Parameters $\alpha \in \mathbb{C}$ ist die folgende Matrix

$$A := \begin{pmatrix} \alpha & 0 & 1/2 & \\ 0 & \alpha & 1/2 & 0 \\ 0 & 1/2 & \alpha & 0 \\ 1/2 & 0 & 0 & \alpha \end{pmatrix} \in \mathbb{C}^{4\times 4},$$

(a) symmetrisch? (b) hermitisch? (c) unitär? (d) normal ($A\overline{A}^T = \overline{A}^T A$)?

Lösung: (a) $\forall \alpha \in \mathbb{C}$ (b) $\alpha \in \mathbb{R}$ (c) $\alpha \pm \frac{\sqrt{3}}{2} i$ (d) $\forall \alpha \in \mathbb{C}$

MAT141 HS18 Serie 10 Aufgabe 2:

Exercise 2 (Hermitian and Unitary Matrices)
Which of the matrices below are unitary? Which ones are hermitian? Write down the inverses of the ones which are unitary.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{2i}{3} & \frac{2i}{3} & \frac{i}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{3\sqrt{2}} & -\frac{i}{3\sqrt{2}} & \frac{4i}{3\sqrt{2}} \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

MAT141 HS18 Serie 11 Aufgabe 2:

For each of the matrices below, check whether it is Hermitian or Unitary. If it is find the eigenvalues and an orthonormal basis of eigenvectors of $\mathbb{C}^n$.

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2+i \\ 0 & 2-i & 1 \end{pmatrix}$$

Lösung:

A is hermitian but not unitary with ONB $\{\left(\frac{-i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T, \left(\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T\}$

B is unitary but not hermitian with ONB $\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T, \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T\}$

C is hermitian but not unitary with ONB $\{(1,0,0)^T, \left(0, \frac{2+i}{\sqrt{10}}, \frac{\sqrt{5}}{\sqrt{10}}\right)^T, \left(0, \frac{-2-i}{\sqrt{10}}, \frac{1}{\sqrt{2}}\right)^T\}$

Lösung HS17 Serie 10 Aufgabe 3

Solution. For which values of $\alpha \in \mathbb{C}$ is the matrix A

1. symmetric? A symmetric matrix defined as a square matrix that is equal to its transpose: $A = A^T$. Since the complex parameter α only occupies diagonal elements, A is a symmetric matrix $\forall \alpha \in \mathbb{C}$.

2. hermitian? A hermitian matrix is a complex square matrix that is equal to its own conjugate transpose: $A = A^*$. In our case, we need to verify if:

$$\begin{pmatrix} \alpha & 0 & 0 & 1/2 \\ 0 & \alpha & 1/2 & 0 \\ 0 & 1/2 & \alpha & 0 \\ 1/2 & 0 & 0 & \alpha \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \bar{\alpha} & 0 & 0 & 1/2 \\ 0 & \bar{\alpha} & 1/2 & 0 \\ 0 & 1/2 & \bar{\alpha} & 0 \\ 1/2 & 0 & 0 & \bar{\alpha} \end{pmatrix}$$

The equality between A and its conjugate transpose is only given if, and only if $\alpha = \bar{\alpha}$, where $\bar{\alpha}$ is the conjugate transpose of α. This equality is only given when α is real.

3. unitary? A unitary matrix A is a complex square matrix whose conjugate transpose is also its inverse: $A^* = A^{-1}$, so that $AA^* = A^*A = \mathbb{I}$. In our case we need to verify if:

$$\begin{pmatrix} |\alpha|^2 + \frac{1}{4} & 0 & 0 & \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} \\ 0 & |\alpha|^2 + \frac{1}{4} & \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} & 0 \\ 0 & \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} & |\alpha|^2 + \frac{1}{4} & 0 \\ \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} & 0 & 0 & |\alpha|^2 + \frac{1}{4} \end{pmatrix} \stackrel{?}{=} \mathbb{I}$$

The equality is only given if two conditions are met. First, the non-diagonal elements must vanish: $\frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha}$. This is only given for values of α without a real part: $\alpha \in \mathbb{C} \notin \mathbb{R}$, so that $\frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} = \frac{1}{2}\alpha - \frac{1}{2}\alpha = 0$. Second, the diagonal elements must equal unity:

$$|\alpha|^2 + \frac{1}{4} = 1 \Leftrightarrow |\alpha|^2 = \frac{3}{4} \Leftrightarrow \alpha = \pm\frac{\sqrt{3}}{2}i.$$

4. normal? A normal matrix A is a complex square matrix which commutes with its conjugate transpose A^*: $AA^* = A^*A$. In our case we need to verify if

$$\begin{pmatrix} \alpha & 0 & 0 & 1/2 \\ 0 & \alpha & 1/2 & 0 \\ 0 & 1/2 & \alpha & 0 \\ 1/2 & 0 & 0 & \alpha \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \bar{\alpha} & 0 & 0 & 1/2 \\ 0 & \bar{\alpha} & 1/2 & 0 \\ 0 & 1/2 & \bar{\alpha} & 0 \\ 1/2 & 0 & 0 & \bar{\alpha} \end{pmatrix}$$

$$\begin{pmatrix} |\alpha|^2 + \frac{1}{4} & 0 & 0 & \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} \\ 0 & |\alpha|^2 + \frac{1}{4} & \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} & 0 \\ 0 & \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} & |\alpha|^2 + \frac{1}{4} & 0 \\ \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} & 0 & 0 & |\alpha|^2 + \frac{1}{4} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} |\alpha|^2 + \frac{1}{4} & 0 & 0 & \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha} \\ 0 & |\alpha|^2 + \frac{1}{4} & \frac{1}{2}\bar{\alpha} + \frac{1}{2}\alpha & 0 \\ 0 & \frac{1}{2}\bar{\alpha} + \frac{1}{2}\alpha & |\alpha|^2 + \frac{1}{4} & 0 \\ \frac{1}{2}\bar{\alpha} + \frac{1}{2}\alpha & 0 & 0 & |\alpha|^2 + \frac{1}{4} \end{pmatrix} \blacksquare$$

The equality is trivial, as the non-diagonal matrix elements are sums of the same terms. Thus, the matrix A is normal $\forall \alpha \in \mathbb{C}$.

Lösung HS18 Serie 10 Aufgabe 2

Exercise 2 (Hermitian and Unitary Matrices)
Which of the matrices below are unitary? Which ones are hermitian? Write down the inverses of the ones which are unitary.

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
B &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
C &= \begin{pmatrix} \frac{2i}{3} & \frac{2i}{3} & \frac{i}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{3\sqrt{2}} & -\frac{i}{3\sqrt{2}} & \frac{4i}{3\sqrt{2}} \end{pmatrix} \\
D &= \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}
\end{aligned}
$$

Solution: There are several ways to check whether a matrix M is unitary. One of them is to check whether its columns form an orthonormal basis of $\mathbb{C}^n$, with respect to the standard inner product

$$\langle v, w\rangle = v_1\bar{w}_1 + \ldots + v_n\bar{w}_n.$$

If this is the case then $M^{-1} = \overline{M}^T$.
On the other hand, a matrix is called *Hermitian* if $\overline{M}^T = M$.

- A has only real entries so $\overline{A}^T = A^T$. Also, we have that $A^T = A$. Hence A is Hermitian. But A is not unitary, since its columns do not form a basis (A is not invertible).
- B is not Hermitian: We have
$$\overline{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \neq B^T$$
(B is however symmetric: $B^T = B$).

  We claim that B is unitary. Denote b_1, b_2 the columns of b. Again, we check the condition (1):
$$
\begin{aligned}
\langle b_1, b_1\rangle &= \frac{1}{2}(1 + i(-i)) = 1, \\
\langle b_1, b_2\rangle &= \frac{1}{2}(-i + i) = 0, \\
\langle b_2, b_2\rangle &= \frac{1}{2}(i(-i) + 1) = 1.
\end{aligned}
$$
  Hence B is unitary, with inverse
$$B^{-1} = \overline{B}^T = \overline{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

- We claim that C is unitary. Denote c_1, c_2, c_3 the columns of C. Again we check the condition (1):
$$
\begin{aligned}
\langle c_1, c_1\rangle &= \frac{2i}{3}\left(-\frac{2i}{3}\right) + \frac{2i}{3}\left(-\frac{2i}{3}\right) + \frac{i}{3}\left(-\frac{i}{3}\right) = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1, \\
\langle c_1, c_2\rangle &= \frac{1}{3\sqrt{2}}(2i - 2i) = 0, \\
\langle c_1, c_3\rangle &= \frac{1}{9\sqrt{2}}\left(2i\cdot i + 2i\cdot i + i\cdot(-4i)\right) = \frac{1}{9\sqrt{2}}(-2 - 2 + 4) = 0, \\
\langle c_2, c_2\rangle &= \frac{1}{2}(1 + 1) = 1, \\
\langle c_2, c_3\rangle &= \frac{1}{12}(-i + i) = 0, \\
\langle c_3, c_3\rangle &= \frac{1}{18}(-i\cdot i + -i\cdot i + 4i\cdot(-4i)) = \frac{1}{18}(1 + 1 + 16) = 1.
\end{aligned}
$$
  Hence C is unitary and its inverse is
$$C^{-1} = \overline{C}^T = \begin{pmatrix} -\frac{2i}{3} & \frac{1}{\sqrt{2}} & \frac{i}{3\sqrt{2}} \\ -\frac{2i}{3} & -\frac{1}{\sqrt{2}} & \frac{i}{3\sqrt{2}} \\ -\frac{i}{3} & 0 & \frac{-4i}{3\sqrt{2}} \end{pmatrix}$$
  Notice that $\overline{C}^T \neq C$, hence C is not Hermitian.

- We have
$$\overline{D}^T = \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \neq D.$$
  Hence D is not Hermitian. It is also not unitary since its columns do not form an orthonormal basis of $\mathbb{C}^3$: E.g. the first and third vector are not orthogonal to each other.

Lösung HS18 Serie 11 Aufgabe 2

Exercise 2 (Unitary and Hermitian Matrices)
For each of the matrices below, check whether it is Hermitian or Unitary. If it is find the eigenvalues and an orthonormal basis of eigenvectors of $\mathbb{C}^n$.

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2+i \\ 0 & 2-i & 1 \end{pmatrix}$$

Solution:

- The matrix A is not unitary, since its column vectors are not normalized (e.g. the first column has norm $\sqrt{2}$. However, it is Hermitian: The transpose is
$$A^T = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \overline{A}$$
and hence $\overline{A}^T = \overline{\overline{A}} = A$. We proceed to compute the orthonormal basis of eigenvectors. The characteristic polynomial is
$$P_A(z) = \det(A - zI_2) = \det \begin{pmatrix} 1-z & i \\ -i & 1-z \end{pmatrix} = (1-z)^2 - 1 = z^2 - 2z = z(z-2)$$
and the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. The eigenspace of $\lambda_1 = 0$ is given by
$$\ker(A - 0I_2) = \ker \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \{(x_1, x_2)^T \in \mathbb{C}^2 | x_1 + ix_2 = 0\} = \{(-ix_2, x_2)^T | x_2 \in \mathbb{C}\}$$
which is spanned by the eigenvector $(-i, 1)^T$. Normalizing it we obtain $v_1 = \frac{1}{\sqrt{2}}(-i, 1)^T$. The eigenspace to $\lambda_2 = 2$ is given by
$$\ker(A - 2I_2) = \ker \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} = \{(x_1, x_2) \in \mathbb{C}^2 | -x_1 + ix_2 = 0\} = \{(ix_2, x_2) | x_2 \in \mathbb{C}\}$$
which is spanned by the eigenvector $(i, 1)^T$. Normalizing we obtain $v_2 = \frac{1}{\sqrt{2}}(i, 1)$ and v_1, v_2 is the desired basis of eigenvectors.

Lösung HS18 Serie 11 Aufgabe 2 - Fortsetzung

- The matrix B is unitary, since its column vectors are normalized and orthognal to each other. However, it is not Hermitian: The transpose is

$$B^T = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = B$$

and hence B is not Hermitian. We proceed to compute the orthonormal basis of eigenvectors. The characteristic polynomial is

$$P_B(z) = \det(B - zI_2) = \det \begin{pmatrix} -z & i \\ i & -z \end{pmatrix} = z^2 + 1$$

and the eigenvalues are $\lambda_1 = +i$ and $\lambda_2 = -i$. The eigenspace of $\lambda_1 = i$ is given by

$$\ker(A - iI_2) = \ker \begin{pmatrix} -i & i \\ i & -i \end{pmatrix} = \{(x_1, x_2)^T \in \mathbb{C}^2 | -ix_1 + ix_2 = 0\} = \{(x_2, x_2)^T | x_2 \in \mathbb{C}\}$$

which is spanned by the eigenvector $(1,1)^T$. Normalizing it we obtain $v_1 = \frac{1}{\sqrt{2}}(1,1)^T$. The eigenspace to $\lambda_2 = 2$ is given by

$$\ker(A + iI_2) = \ker \begin{pmatrix} i & i \\ i & i \end{pmatrix} = \{(x_1, x_2) \in \mathbb{C}^2 | ix_1 + ix_2 = 0\} = \{(-x_2, x_2) | x_2 \in \mathbb{C}\}$$

which is spanned by the eigenvector $(-1,1)^T$. Normalizing we obtain $v_2 = \frac{1}{\sqrt{2}}(-1,1)$ and v_1, v_2 is the desired basis of eigenvectors.

- The matrix C is not unitary, since its column vectors are not normalized (e.g. the first column has norm 5). However, it is Hermitian: The transpose is

$$C^T = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2-i \\ 0 & 2+i & 1 \end{pmatrix} = \overline{C}$$

and hence $\overline{C}^T = \overline{\overline{C}} = C$. We proceed to compute the orthonormal basis of eigenvectors. The characteristic polynomial is

$$P_C(z) = \det(C - zI_2) = \det \begin{pmatrix} 5-z & 0 & 0 \\ 0 & 1-z & 2+i \\ 0 & 2-i & 1-z \end{pmatrix} = (5-z)[(1-z)^2 - (2-i)(2+i)] = (5-z)[z^2 - 2z - 4]$$

and the eigenvalues are $\lambda_1 = 5$ and

$$\lambda_{2,3} = \frac{2 \pm \sqrt{4+16}}{2} = 1 \pm \sqrt{5}$$

i.e.$\lambda_2 = 1 + \sqrt{5}$ and $\lambda_3 = 1 - \sqrt{5}$. The eigenspace of $\lambda_1 = 5$ is given by

$$\ker(A - 5I_2) = \ker \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 2+i \\ 0 & 2-i & -4 \end{pmatrix} = \{(x_1, 0, 0) | x_1 \in \mathbb{C}\}.$$

This can be checked either by solving the corresponding system of linear equation or by noticing that $v_1 = (1,0,0)^T$ lies in this kernel, but this kernel has dimension 1 since it is the eigenspace of an eigenvalue with algebraic multiplicity 1. v_1 is already normalized.
The eigenspace to $\lambda_2 = 1 + \sqrt{5}$ is given by

$$\ker(A - (1 + \sqrt{5}I_2) = \ker \begin{pmatrix} 4-\sqrt{5} & 0 & 0 \\ 0 & -\sqrt{5} & 2+i \\ 0 & 2-i & -\sqrt{5} \end{pmatrix}.$$

Lösung HS18 Serie 11 Aufgabe 2 - Fortsetzung

Writing out the corresponding system of linear equations, the first equation implies $x_1 = 0$. The second equation implies that $(2+i)x_3 = \sqrt{5}x_2$, and hence $x_2 = \frac{2+i}{\sqrt{5}}x_3$. Since we know that the eigenspace has dimension 1, it is given by

$$\{(0, \frac{2+i}{\sqrt{5}}x_3, x_3) | x_3 \in \mathbb{C}\}.$$

Choosing $x_3 = \sqrt{5}$ we obtain the eigenvector $\tilde{v}_2 = (0, 2+i, \sqrt{5})$ which has norm square

$$||\tilde{v}_2||^2 = (2+i)(2-i) + \sqrt{5}^2 = 10.$$

Normalizing we obtain

$$v_2 = \left(0, \frac{2+i}{\sqrt{10}}, \frac{1}{\sqrt{2}}\right)^T.$$

In exactly the same way, we obtain

$$v_3 = \left(0, -\frac{2+i}{\sqrt{10}}, \frac{1}{\sqrt{2}}\right)^T$$

as a normalized eigenvector to λ_3 and v_1, v_2, v_3 is the desired basis of eigenvectors.

DGL-Systeme mit komplexen Zahlen

HS20 Übungsserie 13 Aufgabe 3:

Given the following system of linear ODE:

$$\begin{aligned} y_1' &= -2y_1 + 5y_2, \\ y_2' &= -y_1 - 4y_2, \end{aligned}$$

1. Give the general complex solution.
2. Give the general real solution (obtained from your result in the previous subquestion).

Lösung:

1. $y(t) = C_1 \cdot e^{(-3+2i)t} \begin{pmatrix} -5 \\ 1-2i \end{pmatrix} + C_2 \cdot e^{(-3-2i)t} \begin{pmatrix} -5 \\ 1+2i \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}$
2. $y(t) = C_1 \cdot e^{-3t} \begin{pmatrix} -5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} + C_2 \cdot e^{-3t} \begin{pmatrix} -5\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}$

HS17 Probeprüfung Aufgabe 5:

Betrachten Sie das folgende System von Differentialgleichungen:

$$\begin{aligned} y_1'(t) &= y_1(t) - y_2(t), \\ y_2'(t) &= 2y_1(t) - y_2(t), \end{aligned}$$

a) Bestimmen Sie die allgemeine Lösung des Differentialgleichungssystems.

b) Finden Sie die Lösung mit Anfangswerten $y_1(0) = 1$ und $y_2(0) = 2$.

Lösung:

1. $y(t) = C_1 \begin{pmatrix} \cos(t) \\ \cos(t) + \sin(t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(t) \\ \sin(t) - \cos(t) \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}$
2. $y(t) = 1 \cdot \begin{pmatrix} \cos(t) \\ \cos(t) + \sin(t) \end{pmatrix} - 1 \cdot \begin{pmatrix} \sin(t) \\ \sin(t) - \cos(t) \end{pmatrix} = \begin{pmatrix} \cos(t) - \sin(t) \\ 2\cos(t) \end{pmatrix}$

HS19 Übungsserie 13 Aufgabe 3:

Given the following system of linear ODE:

$$\begin{aligned} y_1' &= y_1 - 2y_2, \\ y_2' &= 2y_1 - y_2, \end{aligned}$$

1. Give the general complex solution.
2. Give the general real solution (obtained from your result in the previous subquestion).

Lösung:

1. $y(t) = C_1 \cdot e^{i\sqrt{3}t} \begin{pmatrix} 2 \\ 1 - i\sqrt{3} \end{pmatrix} + C_2 \cdot e^{-i\sqrt{3}t} \begin{pmatrix} 2 \\ 1 + i\sqrt{3} \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}$
2. $y(t) = C_1 \begin{pmatrix} 2\cos(\sqrt{3}t) \\ \cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \end{pmatrix} + C_2 \begin{pmatrix} 2\sin(\sqrt{3}t) \\ \sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}$

Lösung HS20 - Serie 13 - Aufgabe 3

Exercise 3. Given the following system of linear ODE:

$$y_1' = -2y_1 + 5y_2$$
$$y_2' = -y_1 - 4y_2$$

1. Give the general complex solution.
2. Give the general real solution (obtained from your result in the previous subquestion).

Solution.

1. As usual, we first calculate the eigenvalues of the associated matrix $\begin{pmatrix} -2 & 5 \\ -1 & -4 \end{pmatrix}$ and find the two eigenvalues $\lambda_1 = -3 + 2i$ and $\lambda_2 = -3 - 2i$. Next we look for an eigenvectors to λ_1.

$$A - (-3 + 2i)\mathrm{Id} = \begin{pmatrix} -2 - (-3 + 2i) & 5 \\ -1 & -4 - (-3 + 2i) \end{pmatrix} = \begin{pmatrix} 1 - 2i & 5 \\ -1 & -1 - 2i \end{pmatrix}.$$

Hence, $v_1 = (y_1, y_2)$ is an eigenvector to λ_1 if $(1-2i)y_1 + 5y_2 = 0$, so we find the eigenvector $v_1 = (-5, 1 - 2i)$. To find an eigenvalue to λ_2, we can complex conjugate v_1 and find $v_2 = (-5, 1 + 2i)$. Thus, the general complex solution is given by

$$y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 = C_1 e^{(-3+2i)t} \begin{pmatrix} -5 \\ 1 - 2i \end{pmatrix} + C_2 e^{(-3-2i)t} \begin{pmatrix} -5 \\ 1 + 2i \end{pmatrix}$$

for some $C_1, C_2 \in \mathbb{R}$.

Lösung HS20 - Serie 13 - Aufgabe 3 - Fortsetzung

2. We already have a basis or the space of solutions

$$\left[e^{(-3+2i)t}\begin{pmatrix}-5\\1-2i\end{pmatrix}, e^{(-3-2i)t}\begin{pmatrix}-5\\1+2i\end{pmatrix}\right].$$

We can obtain a different basis with only real entries by using Euler's formula, i.e.

$$\begin{aligned} e^{\lambda_1 t} &= e^{(-3+2i)t} = e^{-3t}e^{2ti} = e^{-3t}(\cos(2t) + i\sin(2t)), \\ e^{\lambda_2 t} &= e^{(-3-2i)t} = e^{-3t}e^{-2ti} = e^{-3t}(\cos(-2t) + i\sin(-2t)) = e^{-3t}(\cos(2t) - i\sin(2t)). \end{aligned}$$

Here, we also used the fact that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. We then obtain a real basis for the space of solutions by combining the two in the following way.

$$[w_1, w_2] := \left[\frac{1}{2}\left(e^{\lambda_1 t}v_1 + e^{\lambda_2 t}v_2\right), \quad \frac{1}{2i}\left(e^{\lambda_1 t}v_1 - e^{\lambda_2 t}v_2\right)\right].$$

We compute

$$\begin{aligned} w_1 &= \frac{1}{2}\left(e^{\lambda_1 t}v_1 + e^{\lambda_2 t}v_2\right) \\ &= \frac{1}{2}\left(e^{-3t}(\cos(2t) + i\sin(2t))\begin{pmatrix}-5\\1-2i\end{pmatrix} + e^{-3t}(\cos(2t) - i\sin(2t))\begin{pmatrix}-5\\1+2i\end{pmatrix}\right) \\ &= \frac{1}{2}e^{-3t}\left(\begin{pmatrix}-5(\cos(2t) + i\sin(2t))\\(1-2i)(\cos(2t) + i\sin(2t))\end{pmatrix} + \begin{pmatrix}-5(\cos(2t) - i\sin(2t))\\(1+2i)(\cos(2t) - i\sin(2t))\end{pmatrix}\right) \\ &= e^{-3t}\begin{pmatrix}-5\cos(2t)\\\cos(2t) + 2\sin(2t)\end{pmatrix}, \\ w_2 &= \frac{1}{2i}\left(e^{\lambda_1 t}v_1 - e^{\lambda_2 t}v_2\right) \\ &= \frac{1}{2i}\left(e^{-3t}(\cos(2t) + i\sin(2t))\begin{pmatrix}-5\\1-2i\end{pmatrix} - e^{-3t}(\cos(2t) - i\sin(2t))\begin{pmatrix}-5\\1+2i\end{pmatrix}\right) \\ &= \frac{1}{2i}e^{-3t}\left(\begin{pmatrix}-5(\cos(2t) + i\sin(2t))\\(1-2i)(\cos(2t) + i\sin(2t))\end{pmatrix} - \begin{pmatrix}-5(\cos(2t) - i\sin(2t))\\(1+2i)(\cos(2t) - i\sin(2t))\end{pmatrix}\right) \\ &= e^{-3t}\begin{pmatrix}-5\sin(2t)\\\sin(2t) - 2\cos(2t)\end{pmatrix}. \end{aligned}$$

The general real solution to the system is thus given by

$$y(t) = C_1 e^{-3t}\begin{pmatrix}-5\cos(2t)\\\cos(2t) + 2\sin(2t)\end{pmatrix} + C_2 e^{-3t}\begin{pmatrix}-5\sin(2t)\\\sin(2t) - 2\cos(2t)\end{pmatrix} \qquad \blacksquare$$

for some $C_1, C_2 \in \mathbb{R}$.

Lösung HS17 - Probeprüfung - Aufgabe 5

Aufgabe 5.

Betrachten Sie das folgende System von Differentialgleichungen:

$$\begin{cases} y_1'(t) &= y_1(t) - y_2(t) \ , \\ y_2'(t) &= 2y_1(t) - y_2(t) \ . \end{cases}$$

(a) Bestimmen Sie die allgemeine Lösung des Differentialgleichungssystems.

(b) Finden Sie die Lösung mit Anfangswerten $y_1(0) = 1$ und $y_2(0) = 2$.

Aufgabe 5 (a) We first compute the eigenvalues and eigenvectors. The characteristic polynom is z^2+1, so the two eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$.

An eigenvector for the eigenvalue $\lambda_1 = i$ is $\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$, and therefore a complex solution is

$$t \mapsto e^{it}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} \cos t + i \sin t \\ \cos t + \sin t + i(\sin t - \cos t) \end{pmatrix}$$

We deduce that the general solution to the system is

$$y_1(t) = a\cos t + b \sin t \qquad a, b \in \mathbb{R}.$$

$$y_2(t) = a(\cos t + \sin t) + b(\sin t - \cos t) = (a-b)\cos t + (b+a)\sin t$$

(b) The conditions $y_1(0) = 1$ and $y_2(0) = 2$ yield the system $\begin{cases} a = 1 \\ a - b = 2 \end{cases}$

that is $a = 1$, $b = -1$.

The solution to the initial value problem is thus

$$y_1(t) = \cos t - \sin t \ , \quad y_2(t) = 2\cos t$$

Lösung HS19 - Serie 13 - Aufgabe 3

Exercise 3 (6 points). Given the following system of linear ODE:

$$\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= 2y_1 - y_2 \end{aligned}$$

1. Give the general complex solution.
2. Give the general real solution (obtained from your result in the previous subquestion).

Solution.

1. As usual, we first calculate the eigenvalues of the associated matrix:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

and find two complex solutions: $\lambda_1 = i\sqrt{3}$ and $\lambda_2 = \overline{\lambda_1} = -i\sqrt{3}$. Next we look for the eigenvectors: For λ_1:

$$A - \lambda_1 1_{\mathbb{R}^2} = \begin{pmatrix} 1 - i\sqrt{3} & -2 \\ 2 & -1 - i\sqrt{3} \end{pmatrix}$$

The first row gives us $(1 - i\sqrt{3})y_1 - 2y_2 = 0$, so we find the eigenvector $v_1 = (2, 1 - i\sqrt{3})$. Complex conjugating gives us the other eigenvector: $v_2 = (2, 1 + i\sqrt{3})$. Thus, the general complex solution is given by:

$$y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 = C_1 e^{i\sqrt{3}t} \begin{pmatrix} 2 \\ 1 - i\sqrt{3} \end{pmatrix} + C_2 e^{-i\sqrt{3}t} \begin{pmatrix} 2 \\ 1 + i\sqrt{3} \end{pmatrix}$$

2. The general linear combination of the coordinatewise real and imaginary parts of $e^{\lambda_1 t} v_1$ (or equivalently those of $e^{\lambda_2 t} v_2$, which is after all just the conjugate of the former) constitutes the general real solution. In order to determine the parts, we first write $e^{\lambda_1 t} = e^{i\sqrt{3}t} = \cos(\sqrt{3}t) + i\sin(\sqrt{3}t)$ using Euler's formula, and calculate

$$\Re(e^{\lambda_1 t} v_1) = \begin{pmatrix} 2\cos(\sqrt{3}t) \\ \cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \end{pmatrix} \quad \text{and} \quad \Im(e^{\lambda_1 t} v_1) = \begin{pmatrix} 2\sin(\sqrt{3}t) \\ -\sqrt{3}\cos(\sqrt{3}t) + \sin(\sqrt{3}t) \end{pmatrix}.$$

Thus, the general real solution is

$$y(t) = C_1 \begin{pmatrix} 2\cos(\sqrt{3}t) \\ \cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \end{pmatrix} + C_2 \begin{pmatrix} 2\sin(\sqrt{3}t) \\ -\sqrt{3}\cos(\sqrt{3}t) + \sin(\sqrt{3}t) \end{pmatrix} \qquad \blacksquare$$

with real C_1 and C_2.