Zusatzaufgaben mit komplexen Zahlen

HS18 Uebungen Serie 5 Aufgabe 2

(a) Geben Sie die Polarform folgender komplexer Zahlen an:

$z_1 = 1 - i, \qquad z_2 = \sqrt{3} - i, \qquad z_3 = 2, \qquad z_4 = -i, \qquad z_5 = 2 + 2\sqrt{3}i.$

(b) Berechnen Sie folgende Multiplikationen und Divisionen mittels Polarform. Geben Sie die Resultate in kartesischer Form an!

$z_1 z_4, \qquad \frac{z_5}{z_2}, \qquad z_2 z_5, \qquad \frac{z_1}{z_4}.$

Lösung:

(a) $z_1 = \sqrt{2}e^{i\cdot 3\pi/4} \quad z_2 = 2e^{-i\cdot\pi/6} \quad z_3 = 2e^{i\cdot 0} \quad z_4 = 1e^{i\cdot 3\pi/2} \quad z_5 = 4e^{i\cdot\pi/3}$

(b) $z_1 z_4 = 1 + i \quad \frac{z_5}{z_2} = 2i \quad z_2 z_5 = 4\sqrt{3} + 4i \quad \frac{z_1}{z_4} = -1 - i$

HS18 Uebungen Serie 6 Aufgabe 4a)

Löse folgendes komplexe lineare Gleichungssystem:

(i)
$$\begin{aligned} z_1 + iz_2 &= 2 \\ (1+i)z_1 + (i-1)z_2 &= 2i + 2 \end{aligned}$$

(ii)
$$\begin{aligned} iz_1 + z_2 &= 1 \\ z_1 + iz_2 &= i \end{aligned}$$

(iii)
$$\begin{aligned} 2z_1 - (2+i)z_2 + z_3 &= i \\ 4z_1 - iz_3 &= 1 \end{aligned}$$

(iv)
$$\begin{aligned} z_1 + iz_2 &= 0 \\ 4iz_2 - 7z_3 &= 0 \\ -5z_2 - 9iz_3 &= 1 \end{aligned}$$

Lösung:

(i) $L = \{(2 - iz_2, z_2),\ z_2 \in \mathbb{C}\}$ (ii) $L = \{(0, 1)\}$

(iii) $L = \{(\frac{1+z_3}{4}, \frac{1-2i+3z_3}{4+2i}), z_3\ z_3 \in \mathbb{C}\}$ (iv) $L = \{(7i, -7, -4i)\}$

HS18 Uebungen Serie 6 Aufgabe 4b

Berechne die Determinante für folgende Matrizen. Gib an, ob sie regulär sind und falls ja, berechne die Inverse.

$$A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \qquad B = \begin{pmatrix} 1+i & 1 \\ i & 1-i \end{pmatrix} \qquad C = \begin{pmatrix} 1 & i & 2 \\ 2 & 1+2i & 7 \\ 1 & 1+i & 6 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & i & 2 \\ 2 & 1+2i & 8 \\ 2 & 1+i & 6 \end{pmatrix}$$

Lösung:

$\det(A) = 0 \quad \det(B) = 2 - i \rightarrow B^{-1} = \frac{1}{2-i}\begin{pmatrix} 1-i & -1 \\ -i & 1+i \end{pmatrix}$

$\det(C) = 1 \rightarrow C^{-1} = \begin{pmatrix} -1+5i & 2-4i & -2+3i \\ -5 & 4 & -3 \\ 1 & -1 & 1 \end{pmatrix} \quad \det(D) = 0$

HS17 Uebungen Serie 7 Aufgabe 1

Entscheide, ob folgende Vektoren in $\mathbb{C}^2$ eine Basis von $\mathbb{C}^2$ bilden:

(a)

$$v^{(1)} = \begin{pmatrix} 1+i \\ -1+i \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} -2+2i \\ -2-2i \end{pmatrix}$$

(b)

$$v^{(1)} = \begin{pmatrix} 1+i \\ 2-i \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} -i \\ 2+2i \end{pmatrix}$$

Lösung: (a) $\det(v^{(1)}, v^{(2)}) = 0 \to$ linear abhängig $\Rightarrow$ keine Basis
(b) $\det(v^{(1)}, v^{(2)}) = 1 + 6i \neq 0 \to$ linear unabhängig $\Rightarrow$ Basis

HS17 Uebungen Serie 6 Aufgabe 4

(a) Für welche Werte von $a \in \mathbb{C}$ hat das folgende komplexe lineare Gleichungssystem

(i) keine Lösung? (ii) genau eine Lösung? (iii) unendlich viele Lösungen?

$$\begin{aligned} z_1 + iz_2 + (1-i)z_3 &= 0 \\ ia^2 z_1 + z_2 + (1+i)a^2 z_3 &= a - i \\ iz_1 - z_2 + iz_3 &= a + i \end{aligned}$$

(b) Lösen Sie das Gleichungssystem für $a = 3i$.

Lösung: (a) (i) $a = -i$ (ii) $a \neq \pm i$ (iii) $a = +i$ (b) $L = \{(15/4 + 4i, -i/4, -4i)\}$

HS15 Uebungen Serie 6 Aufgabe 1

a) Entscheide, ob folgende Vektoren in $\mathbb{C}^3$ linear abhängig sind oder nicht:

$$a^{(1)} = \begin{bmatrix} 1+i \\ 1-i \\ -1+i \end{bmatrix}, \qquad a^{(2)} = \begin{bmatrix} 0 \\ 4-2i \\ -3+5i \end{bmatrix}, \qquad a^{(3)} = \begin{bmatrix} 1+i \\ 0 \\ 3 \end{bmatrix}.$$

b) Für welche komplexen Zahlen $z \in \mathbb{C}$ verschwindet die Determinante von

$$\begin{bmatrix} 1+2i & 3+4i \\ z & 1-2i \end{bmatrix}^{25}$$

Lösung: a) linear unabhängig b) $z = \frac{3}{5} - \frac{4}{5}i$

HS15 Probeprüfung Aufgabe 2

i) Berechne die Determinante von

$$\begin{pmatrix} i & 0 & 1 \\ -1 & 1 & -1 \\ 1+i & -1+i & 1-i \end{pmatrix}^4.$$

iii) Bestimme die Nullstellen von $p(z) = z^4 - i$.

Lösung: i) 16 iii) $e^{\pi/8+(j-1)\cdot\pi/2},\ 1 \leq j \leq 4$

HS15 Uebungen Serie 10 Aufgabe 1

Sei $A = \begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix} \in \mathbb{C}^{2x2}$.

i) Verifiziere, dass A hermitesch ist.

ii) Bestimme die Eigenwerte von A.

iii) Finde eine unitäre Matrix S so dass $S^{-1}AS$ eine Diagonalmatrix ist.

Lösung: i) $\overline{A^T} = A$ ii) $\lambda_1 = -1, \lambda_2 = 3$

iii) $S = \frac{1}{\sqrt{2}}[v^{(1)}\ v^{(2)}]$ mit $v^{(1)} = (-i, 1)^T,\ v^{(2)} = (i, 1)^T$ und $D = \mathrm{diag}(-1, 3)$

HS15 Uebungen Serie 9 Aufgabe 3

i) Bestimme die Eigenwerte und dazugehörigen Eigenvektoren von $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Finde eine Basis $[v] = [v^{(1)}, v^{(2)}]$ von $\mathbb{C}^2$ so dass $(T_A)_{[v]\to[v]}$ eine Diagonalmatrix ist.

ii) Dieselbe Aufgabe wie in i) für die Matrix $B = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}$.

Lösung: i) $\lambda_1 = 1-2i,\ \lambda_2 = 1+2i$ $[v] = [v^{(1)}, v^{(2)}] = [(i,1)^T, (-i,1)^T]$ $D = \begin{pmatrix} 1-2i & 0 \\ 0 & 1+2i \end{pmatrix}$

ii) $z_1 = -1,\ z_2 = 6$ $[v] = [v^{(1)}, v^{(2)}] = [(-\frac{3+i}{2}, 1)^T, (\frac{3+i}{5}, 1)^T]$ $D = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$

Lösungen - Zusatzaufgaben mit komplexen Zahlen

Exercise 2 (Polar form)

(a) Convert the following complex numbers to polar form:

$$z_1 = i - 1, \qquad z_2 = \sqrt{3} - i, \qquad z_3 = 2, \qquad z_4 = -i, \qquad z_5 = 2 + 2\sqrt{3}i.$$

(b) Compute the following multiplications and divisions *using the polar form*. Give the result *in cartesian form*.

$$z_1 z_4, \qquad \frac{z_5}{z_2}, \qquad z_2 z_5, \qquad \frac{z_1}{z_4}.$$

Solution:

(a) We have

$$z_1 = \sqrt{2}e(3\pi/4), \qquad z_2 = 2e(-\pi/6), \qquad z_3 = 2e(0), \qquad z_4 = 1e(3\pi/2), \qquad z_5 = 4e(\pi/3).$$

(b) We have

$$\begin{aligned}
z_1 z_4 &= \sqrt{2}e(3\pi/4)e(3\pi/2) = \sqrt{2}e(9\pi/4) = \sqrt{2}e(\pi/4) = 1 + i, \\
\frac{z_5}{z_2} &= \frac{4e(\pi/3)}{2e(-\pi/6)} = 2e(\pi/3 + \pi/6) = 2e(\pi/6) = 2i, \\
z_2 z_5 &= 2e(-\pi/6)4e(\pi/3) = 8e(\pi/3 - \pi/6) = 4(\sqrt{3} + i), \\
\frac{z_1}{z_4} &= \frac{\sqrt{2}e(3\pi/4)}{e(3\pi/2)} = \sqrt{2}e(3\pi/4 - 3\pi/2) = \sqrt{2}e(-3\pi/4) = -1 - i.
\end{aligned}$$

Exercise 4 (Complex linear systems and matrices)

(a) Give the sets of solutions of the following systems of linear equations[1]:

(i)

$$\begin{cases} z_1 + iz_2 & = 2 \\ (1+i)z_1 + (i-1)z_2 & = 2i+2 \end{cases}$$

(ii)

$$\begin{cases} iz_1 + z_2 & = 1 \\ z_1 + iz_2 & = i \end{cases}$$

(iii)

$$\begin{cases} 2z_1 - (2+i)z_2 + z_3 & = i \\ 4z_1 - z_3 & = 1 \end{cases}$$

(iv)

$$\begin{cases} z_1 + iz_2 & = 0 \\ 4iz_2 - 7z_3 & = 0 \\ -5z_2 - 9iz_3 & = 1 \end{cases}$$

(b) Compute the determinant of the following matrices. Decide whether they are regular. If so, compute the inverse matrix.

$$A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \qquad B = \begin{pmatrix} 1+i & 1 \\ i & 1-i \end{pmatrix} \qquad C = \begin{pmatrix} 1 & i & 2 \\ 2 & 1+2i & 7 \\ 1 & 1+i & 6 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & i & 2 \\ 2 & 1+2i & 8 \\ 2 & 1+i & 6 \end{pmatrix}$$

Solution:

(a) Also for linear systems with complex coefficients we can use the Gauss algorithm as before.

(i) The extended coefficient matrix is

$$\left(\begin{array}{cc|c} 1 & i & 2 \\ (1+i) & (i-1) & 2i+2 \end{array}\right) \overset{}{\underset{II-(1+i)I}{\rightsquigarrow}} \left(\begin{array}{cc|c} 1 & i & 2 \\ 0 & 0 & 0 \end{array}\right)$$

Hence the set of solutions has a free parameter, say z_2, and by equation I we get $z_1 = 2 - iz_2$, so that the set of solutions is

$$L = \{(2 - iz_2, z_2), z_2 \in \mathbb{C}\}.$$

Notice that since we consider complex solutions now, the parameter z_2 can take any *complex value.*

(ii) The extended coefficient matrix is

$$\left(\begin{array}{cc|c} i & 1 & 1 \\ 1 & i & i \end{array}\right) \underset{II+iI}{\rightsquigarrow} \left(\begin{array}{cc|c} i & 1 & 1 \\ 0 & 2i & 2i \end{array}\right)$$

Hence we get $z_2 = i$ and then from the first equation $z_1 = 1$, so the set of solutions is

$$L = \{(0, 1)\}.$$

(iii) The extended coefficient matrix is

$$\left(\begin{array}{ccc|c} 2 & -(2+i) & 1 & i \\ 4 & 0 & -1 & 1 \end{array}\right) \underset{II-2I}{\rightsquigarrow} \left(\begin{array}{ccc|c} 2 & -(2+i) & 1 & i \\ 0 & (4+2i) & -3 & 1-2i \end{array}\right)$$

[1] We now use $z_1, z_2, \ldots$ as variables to emphasize that solutions can be complex

The system has a free parameter, say z_3. Then, we get that

$$z_2 = \frac{1-2i+3z_3}{4+2i}$$

and

$$2z_1 = i + (2+i)z_2 - z_3 = i + (1-2i+3z_3)/2 - z_3 = \frac{1+z_3}{2}$$

so that the set of solutions is

$$L = \left\{ \left(\frac{1+z_3}{4}, \frac{1-2i+3z_3}{4+2i}, z_3 \right), z_3 \in \mathbb{C} \right\}.$$

(iv) The extended coefficient matrix is

$$\left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 4i & -7 & 0 \\ 0 & -5 & -9i & -1 \end{array}\right) \underset{III \to 4iIII}{\overset{II \to 5II}{\rightsquigarrow}} \left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 20i & -35 & 0 \\ 0 & -20i & 36 & -1 \end{array}\right) \overset{III+II}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 20i & -35 & 0 \\ 0 & 0 & 1 & -4i \end{array}\right)$$

which yields the set of solutions

$$L = \{(7i, -7, -4i)\}.$$

(b) We have $\det A = i(-i) - 1 = 0$, hence this matrix is not regular. Then, $\det B = (1+i)(1-i) - i = 2 - i$. Hence, B is invertible, and we can use the usual formula for the inverse and

$$B^{-1} = \frac{1}{2-i} \begin{pmatrix} 1-i & -1 \\ -i & 1+i \end{pmatrix}.$$

For the 3×3 matrices we use row operations. For C, we have

$$C = \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 7 & 0 & 1 & 0 \\ 1 & 1+i & 6 & 0 & 0 & 1 \end{array}\right) \underset{III-I}{\overset{II-2I}{\rightsquigarrow}} \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 1 & 4 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{III-II}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

so C is regular and the determinant of C is 1. Continuing, we get

$$\left(\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \underset{I-2III}{\overset{II-3III}{\rightsquigarrow}} \left(\begin{array}{ccc|ccc} 1 & i & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & -5 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\overset{I-iII}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1+5i & 2-4i & -2+3i \\ 0 & 1 & 0 & -5 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

so the inverse of C is

$$C^{-1} = \begin{pmatrix} -1+5i & 2-4i & -2+3i \\ -5 & 4 & -3 \\ 1 & -1 & 1 \end{pmatrix}.$$

Similarly, for D we have

$$D = \left(\begin{array}{ccc|ccc} 0 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 8 & 0 & 1 & 0 \\ 2 & 1+i & 6 & 0 & 0 & 1 \end{array}\right) \overset{III \leftrightarrow I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 2 & 1+2i & 8 & 0 & 1 & 0 \end{array}\right)$$

$$\overset{III-I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 0 & i & 2 & 0 & 1 & -1 \end{array}\right) \overset{III-II}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 2 & 1+i & 6 & 0 & 0 & 1 \\ 0 & i & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right)$$

hence D is not regular, i.e. $\det(D) = 0$.

Exercise 1 (4 points). Decide if the given vectors in $\mathbb{C}^2$ form a basis of $\mathbb{C}^2$.

1.
$$v^{(1)} = \begin{pmatrix} 1+\mathrm{i} \\ -1+\mathrm{i} \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} -2+2\mathrm{i} \\ -2-2\mathrm{i} \end{pmatrix}.$$

2.
$$v^{(1)} = \begin{pmatrix} 1+\mathrm{i} \\ 2-\mathrm{i} \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} -\mathrm{i} \\ 2+2\mathrm{i} \end{pmatrix}.$$

Solution.

1. $v^{(2)} = 2\mathrm{i} \cdot v^{(1)}$. Because the two vectors are dependent, the do not form a basis.
2. $\det\left(v^{(1)}\ v^{(2)}\right) = 1 + 6\mathrm{i} \neq 0$. Therefore, the two vectors are independent and form a basis.

Exercise 4. (4 points) Consider the following linear system with complex coefficients

$$\begin{aligned} &z_1 + iz_2 + (1-i)z_3 = 0 \\ &ia^2 z_1 + z_2 + (1+i)a^2 z_3 = a - i \\ &iz_1 - z_2 + iz_3 = a + i\,, \end{aligned}$$

where $a \in \mathbb{C}$.

1. Use Gaussian elimination to bring it in row echelon form.(1 point)
2. Discuss the existence and unicity of solutions according to the values of a. (2 points)
3. Solve the system explicitly for $a = 3i$. (1 point)

Solution.

1. By substituting $R_2 \to R_2 - ia^2 R_1$ and $R_3 \to R_3 - iR_1$ we find
$$\left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ ia^2 & 1 & (1+i)a^2 & a-i \\ i & -1 & i & a+i \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ 0 & 1+a^2 & 0 & a-i \\ 0 & 0 & -1 & a+i \end{array}\right).$$
2. For $a^2 + 1 \neq 0 \leftrightarrow a \neq \pm i$ the coefficient matrix is non singular, hence the solution is unique.
For $a = -i$ the system becomes
$$\left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 0 & -1 & 0 \end{array}\right).$$
The second equation reads $0 \cdot z_2 = -2i$, so the system has no solution.
For $a = i$ we have
$$\left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2i \end{array}\right).$$
The second equation gives $0 \cdot z_2 = 0$, thus the system has infinite solutions.
3. For $a = 3i$ the system becomes
$$\left(\begin{array}{ccc|c} 1 & i & 1-i & 0 \\ 0 & -8 & 0 & 2i \\ 0 & 0 & -1 & 4i \end{array}\right)$$
and the solution is given by
$$L = \left\{\left(\frac{15}{4} + 4i, -\frac{i}{4}, -4i\right)\right\}. \qquad \blacksquare$$

HS15 - Serie 6 - Aufgabe 1a)

///// The first step of Gauss elimination yields

$$\begin{bmatrix} 1+\mathrm{i} & 0 & 1+\mathrm{i} \\ 1-\mathrm{i} & 4-2\mathrm{i} & 0 \\ -1+\mathrm{i} & -3+5\mathrm{i} & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4-2\mathrm{i} & -1+\mathrm{i} \\ 0 & -3+5\mathrm{i} & 4-\mathrm{i} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 20-10\mathrm{i} & -5+5\mathrm{i} \\ 0 & -6+10\mathrm{i} & 8-2\mathrm{i} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 14 & 3+3\mathrm{i} \\ 0 & -6+10\mathrm{i} & 8-2\mathrm{i} \end{bmatrix},$$

and we compute

$$8-2\mathrm{i}-\frac{1}{14}(3+3\mathrm{i})(-6+10\mathrm{i}) = 8-2\mathrm{i}-\frac{1}{14}(-48+12\mathrm{i}) \neq 0.$$

Therefore, the three vectors are $\mathbb{C}$-linearly independent.

Alternatively, one can compute the determinant

$$\begin{aligned} &(1+\mathrm{i})(4-2\mathrm{i})3+(1+\mathrm{i})(1-\mathrm{i})(-3+5\mathrm{i})-((-1+\mathrm{i})(4-2\mathrm{i})(1+\mathrm{i})) \\ &= (6+2\mathrm{i})3+2(-3+5\mathrm{i})+2(4-2\mathrm{i}) = (18-6+8)+(6+10-4)\mathrm{i} = 20+12\mathrm{i} \neq 0. \quad /////\end{aligned}$$

HS15 - Serie 6 - Aufgabe 1b)

$$\det B = [(1+2\mathrm{i})(1-2\mathrm{i})-(3+4\mathrm{i})z]^{25} = [5-(3+4\mathrm{i})z]^{25}.$$

Therefore, $\det B = 0$ if

$$z = \frac{5}{3+4\mathrm{i}} = \frac{5(3-4\mathrm{i})}{(3+4\mathrm{i})(3-4\mathrm{i})} = \frac{15-20\mathrm{i}}{25} = \frac{3}{5}-\frac{4}{5}\mathrm{i}. \quad /////$$

HS15 - Probeprüfung - Aufgabe 2a)

(i) Berechne die Determinante von

$$\begin{pmatrix} i & 0 & 1 \\ -1 & 1 & -1 \\ 1+i & -1+i & 1-i \end{pmatrix}^4.$$

///// By expanding $\det(A)$ with respect to the first row one computes $\det(A) = i((1-i)+(-1+i))+1(1-i-1-i) = -2i$ and $\det(A^4) = \det(A)^4 = 16$. /////

HS15 - Probeprüfung - Aufgabe 2c)

(iii) Bestimme die Nullstellen von $p(z) = z^4 - i$.

///// The solutions of $z^4 = i = e^{i\frac{\pi}{2}}$ are given by

$$z_i = e^{i\psi_j},\ j \in \{0,\dots,3\} \quad \text{with} \quad \psi_j = \frac{\pi}{8}+j(\frac{\pi}{2}). \quad /////$$

Aufgabe 1 Sei $A = \begin{bmatrix} 1 & 2\mathrm{i} \\ -2\mathrm{i} & 1 \end{bmatrix} \in \mathbb{C}^{2\times 2}$.

(i) Verifiziere, dass A hermitesch ist.

///// $\overline{A^\top} = \begin{bmatrix} 1 & 2\mathrm{i} \\ -2\mathrm{i} & 1 \end{bmatrix} = A.$ /////

(ii) Bestimme die Eigenwerte von A.

///// We note that $\chi(z) = (1-z)^2 - 4 = z^2 - 2z - 3$ and further compute

$$z_\pm = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2.$$

Hence $\lambda_1 = -1$ and $\lambda_2 = 3$ are the eigenvalues of A. /////

(iii) Finde eine unitäre Matrix S so dass $S^{-1}AS$ eine Diagonalmatrix ist.

///// Since A is hermitean it suffices to compute the eigenvectors. For λ_1

$$\begin{bmatrix} 2 & 2\mathrm{i} \\ -2\mathrm{i} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \mathrm{i} \\ 0 & 0 \end{bmatrix},$$

hence $v^{(1)} = (-\mathrm{i}, 1)$. For λ_2

$$\begin{bmatrix} -2 & 2\mathrm{i} \\ -2\mathrm{i} & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\mathrm{i} \\ 0 & 0 \end{bmatrix},$$

hence $v^{(2)} = (\mathrm{i}, 1)$. Clearly $\|v^{(1)}\| = \|v^{(2)}\| = \sqrt{2}$ so let $S = \frac{1}{\sqrt{2}}[v^{(1)}\ v^{(2)}]$ then

$$S^{-1}AS = S^*AS = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$ /////

Aufgabe 3

(i) Bestimme die Eigenwerte und dazugehörige Eigenvektoren von $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Finde eine Basis $[v] = [v^{(1)}, v^{(2)}]$ von $\mathbb{C}^2$ so dass $(T_A)_{[v]\to[v]}$ eine Diagonalmatrix ist.

///// We compute

$$\det(A - z) = (1-z)^2 + 4 = z^2 - 2z + 5.$$

Consequently, the eigenvalues are given by

$$z_\pm = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2\mathrm{i}.$$

To determine the eigenvectors we solve $(A - z_\pm)x = 0$. That is for z_-

$$\left[\begin{array}{cc|c} 2\mathrm{i} & 2 & 0 \\ -2 & 2\mathrm{i} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2\mathrm{i} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

hence we may choose $v_- = (\mathrm{i}, 1)$. Similarly, for z_+,

$$\left[\begin{array}{cc|c} -2\mathrm{i} & 2 & 0 \\ -2 & -2\mathrm{i} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -2\mathrm{i} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so we may choose $v_+ = (-\mathrm{i}, 1)$. Clearly, v_+ and v_- are $\mathbb{C}$-linearly independent. Let $[v] = [v_-, v_+]$ and note that $Av_\pm = z_\pm v_\pm$ hence

$$(T_A)_{[v]\to[v]} = \begin{bmatrix} 1-2\mathrm{i} & 0 \\ 0 & 1+2\mathrm{i} \end{bmatrix}. \quad /////$$

(ii) Dieselbe Aufgabe wie in (i) für die Matrix $B = \begin{bmatrix} 1 & 3+\mathrm{i} \\ 3-\mathrm{i} & 4 \end{bmatrix}$.

///// We compute the characteristic polynomial to be

$$\det(B - z) = (1-z)(4-z) - (3+\mathrm{i})(3-\mathrm{i}) = z^2 - 5z - 6.$$

Therefore,

$$z_\pm = \frac{5 \pm \sqrt{49}}{2} = \frac{5 \pm 7}{2}.$$

The eigenvalues thus are $z_- = -1$ and $z_+ = 6$. To compute the eigenvectors we solve $(A - z_\pm)x = 0$. So for z_-

$$\left[\begin{array}{cc|c} 2 & 3+\mathrm{i} & 0 \\ 3-\mathrm{i} & 5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3+\mathrm{i} & 0 \\ 2 & 3+\mathrm{i} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3+\mathrm{i} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

hence $v_- = (-\frac{3+\mathrm{i}}{2}, 1)$, and further for z_+

$$\left[\begin{array}{cc|c} -5 & 3+\mathrm{i} & 0 \\ 3-\mathrm{i} & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -5 & 3+\mathrm{i} & 0 \\ -5 & 3+\mathrm{i} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -5 & 3+\mathrm{i} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

we get $v_+ = (\frac{3+\mathrm{i}}{5}, 1)$. With $[v] = [v_-, v_+]$ we obtain

$$(T_B)_{[v]\to[v]} = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}. \quad /////$$